Numerical Methods (PEM2062 & PEM2056)
Tutorial's solutions

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1. $\text{Rel}(p^*) = \left|\dfrac{p - p^*}{p}\right| < 10^{-e}$

(a) $150(1-10^{-3}) < p^* < 150(1+10^{-3})$
$149.85 < p^* < 150.15$

(b) $p\,(1-10^{-4}) < p^* < p\,(1+10^{-4})$
$0.9999p < p^* < 1.0001p$

4. (a) $\begin{Bmatrix} 0.10_2 \\ 0.11_2 \end{Bmatrix} \times \begin{Bmatrix} 2^{00} \\ 2^{01} \\ 2^{02} \\ 2^{03} \end{Bmatrix} = \tfrac{1}{2}, \tfrac{3}{4}, 1, \tfrac{3}{2}, 2, 3, 4, 6.$

(b) $\begin{Bmatrix} 0.100_2 \\ 0.101_2 \\ 0.110_2 \\ 0.111_2 \end{Bmatrix} \times \begin{Bmatrix} 2^{0} \\ 2^{1} \end{Bmatrix} = 1, \tfrac{5}{8}, \tfrac{3}{4}, \tfrac{7}{8}, 1, 1\tfrac{1}{4}, 1\tfrac{1}{2}, 1\tfrac{3}{4}.$

7. The second method is better as it postpones the subtractive cancellation error to the last step of calculation.

8.

x	$y(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$	$\Delta^5 f(x)$	$\Delta^6 f(x)$	$\Delta^7 f(x)$
0	0							
		0						
1	0		0					
		0		1				
2	0		1		–4 + 1 = –3			
		1		–3 +1 = –2		5		
3	1		–2 + 1 = –1		6 + –4 = 2		–5	
		–1+1 = 0		3 + –3 = 0		0		0
4	1		1 + –2 = –1		–4 + 6 = 2		–5	
		–1		–1 + 3 = 2		–5		
5	0		1		1 + –4 = –3			
		0		–1				
6	0		0					
		0						
7	0							

The error influences a triangular portion of the difference table. Each error increases with a binomial coefficient pattern. Since we have adjacent errors, for certain differences, we have the summation of values contributed from each error.

11. For degree = 1 and n = 1. By taking x = 8.3, 8.6,

$$P_1(x) = \frac{(x-8.6)}{(8.3-8.6)}(17.56492) + \frac{(x-8.3)}{(8.6-8.3)}(18.50515)$$

$$P_1(8.4) = 17.87833.$$

Note: take arbitrary 2 data points, as long as point 8.4 is bounded by them on both sides.

For degree=2 and n = 2. By taking x = 8.3, 8.6 and 8.7,

$$P_2(x) = \frac{(x-8.6)(x-8.7)}{(8.3-8.6)(8.3-8.7)}(17.56492) + \frac{(x-8.3)(x-8.7)}{(8.6-8.3)(8.6-8.7)}(18.50515)$$
$$+ \frac{(x-8.3)(x-8.6)}{(8.7-8.3)(8.7-8.6)}(18.2091)$$

$$P_2(8.4) = 17.87716.$$

When degree=2, the error bound is given by

$$x \ln x - P_2(x) = \frac{(x-8.3)(x-8.6)(x-8.7)}{3!} f'''(c_x)$$

for some c_x between the minimum and maximum of 8.3 and 8.7.

Since x = 8.4, the upper and lower bound is

$$\frac{(8.4-8.3)(8.4-8.6)(8.4-8.7)}{6(8.7)^2} \le |8.4\ln(8.4) - P_2(8.4)| \le \frac{(8.4-8.3)(8.4-8.6)(8.4-8.7)}{6(8.3)^2}$$

$$1.321178 \times 10^{-5} \le |8.4\ln(8.4) - P_2(8.4)| \le 1.451589 \times 10^{-5}$$

The Absolute error is $|8.4\ln(8.4) - P_2(8.4)| = 1.3672 \times 10^{-5}$.

Note that the Absolute error is within the error bound.

12. The divided difference table is:

x_i	$f(x_i)$	$f[x_i, x_{i+1}]$	$f[x_i, x_{i+1}, x_{i+2}]$	$f[x_i, x_{i+1}, \ldots, x_{i+3}]$	$f[x_i, x_{i+1}, \ldots, x_{i+4}]$	$f[x_i, x_{i+1}, \ldots, x_{i+5}]$
0	-6					
		1.0517				
0.1	-5.89483		0.5725			
		1.22345		0.215		
0.3	-5.65014		0.7015		0.063016	
		1.5742		0.278016		0.014159
0.6	-5.17788		0.951714		0.078591	
		2.2404		0.356607		
1	-4.28172		1.237			
		2.8589				
1.1	-3.99583					

$P_4(x) = -6 + 1.05170x + 0.57250x(x-0.1) + 0.21500x(x-0.1)(x-0.3)$
$+ 0.063016x(x-0.1)(x-0.3)(x-0.6)$

$P_5(x) = P_4(x) + 0.014159x(x-0.1)(x-0.3)(x-0.6)(x-1)$

13. The divided difference table is:

x_i	$f(x_i)$	$f[x_i, x_{i+1}]$	$f[x_i, x_{i+1}, x_{i+2}]$	$f[x_i, x_{i+1}, x_{i+2}, x_{i+3}]$	$f[x_i, x_{i+1}, x_{i+2}, x_{i+3}, x_{i+4}]$
1	0				
		341			
4	1023		220		
		781		38	
3	242		144		9
		61		47	
-1	-2		50		
		11			
2	31				

$P_4(x) = 341(x-1) + 220(x-1)(x-4) + 38(x-1)(x-4)(x-3)$
$\quad + 9(x-1)(x-4)(x-3)(x+1)$

$P_4(3.2) = 336.3184$

14. The difference table is:

x_i	$f(x_i)$	$\Delta f(x_i)$	$\Delta^2 f(x_i)$	$\Delta^3 f(x_i)$	$\Delta^4 f(x_i)$
0	1				
		0.2214			
0.2	1.2214		0.04902		
		0.27042		0.01086	
0.4	1.49182		0.05988		0.00238
		0.3303		0.01324	
0.6	1.82212		0.07312		
		0.40342			
0.8	2.22554				

Newton forward difference gives

$$P_4(0.05) = f(x_0) + (0.25)\Delta f(x_0) + \frac{0.25(0.25-1)}{2!}\Delta^2 f(x_0) + \frac{0.25(0.25-1)(0.25-2)}{3!}\Delta^3 f(x_0)$$
$$+ \frac{0.25(0.25-1)(0.25-2)(0.25-3)}{4!}\Delta^4 f(x_0)$$
$$= 1.05126$$

Newton backward-difference gives

$$P_4(0.65) = f(x_4) - (0.75)\nabla f(x_4) + \frac{0.75(0.75-1)}{2!}\nabla^2 f(x_4) - \frac{0.75(0.75-1)(0.75-2)}{3!}\nabla^3 f(x_4)$$
$$+ \frac{0.75(0.75-1)(0.75-2)(0.75-3)}{4!}\nabla^4 f(x_4)$$
$$= 1.91555$$

15.

x_i	$f(x_i)$	$f[x_i, x_{i+1}]$	$f[x_i, x_{i+1}, x_{i+2}]$	$f[x_i, x_{i+1}, x_{i+2}, x_{i+3}]$
0	0			
		$2y$		
0.5	y		$6-4y$	
		$2(3-y)$		$(16y-32)/6$
1	3		$2(2y-7)/3$	
		-1		
2	2			

From the divided difference table, $f[x_i, x_{i+1}, x_{i+2}]$ gives the coefficient of x^3.
Hence, $(16y-32)/6 = 6$, $y = 4.25$.

16. For linear interpolating polynomial, the error is

$$\sin(x-2) - P_1(x) = \frac{(x-x_0)(x_1-x)\sin(c_x-2)}{2}$$

with some c_x between the minimum and maximum of x_0 and x_1.

Since $\max\limits_{x_0 \le x \le x_1} \frac{(x-x_0)(x_1-x)}{2} = \frac{h^2}{8}$ and $\max\limits_{\frac{-p}{6} \le c \le \frac{p}{6}} \{\sin(c-2)\} = 1$, the error bound is

$$|\sin(x-2) - P_1(x)| \le \frac{h^2}{8}$$

18. Let $f(x) = \log_{10} x$, $f''(x) = -\frac{\log_{10} e}{x^2}$.

For linear interpolating polynomial, the error is

$$\log_{10} x - P_1(x) = \frac{(x-x_0)(x-x_1)\log_{10} e}{2!\, c_x^{\ 2}}$$

with some c_x between the minimum and maximum of x_0 and x_1.

Since $\max\limits_{x_0 \le x \le x_1} \frac{(x-x_0)(x_1-x)}{2} = \frac{h^2}{8}$ and by taking $c_x = 1$, the error bound is

$$|\log_{10} x - P_1(x)| \le \frac{h^2 \log_{10} e}{8}$$

For $\frac{h^2 \log_{10} e}{8} \le 10^{-6}$,

$$0.0542868h^2 \le 10^{-6}$$

We take h be 4×10^{-3}.

19.

x	$f(x)$	Differences	$\approx f'(x)$	Absolute error	Error bound
0.5	0.4794	Forward	=(0.5646–0.4794)/0.1=0.852	0.025583	0.028232
0.6	0.5646	Forward	=(0.6442–0.5646)/0.1=0.796	0.029336	0.032211
		Backward	=(0.5646–0.4794)/0.1=0.852	0.026664	0.028232
0.7	0.6442	Backward	=(0.6442–0.5646)/0.1=0.796	0.031158	0.032211

20.

The quadratic interpolating polynomial is

$$f(x) = P_2(x) + \frac{(x-x_0)(x-x_1)(x-x_2)}{3!} f^{(3)}(c_x)$$

$$= \frac{(x-x_1)(x-x_2)}{2h^2} f(x_0) - \frac{(x-x_0)(x-x_2)}{h^2} f(x_1) + \frac{(x-x_0)(x-x_1)}{2h^2} f(x_2)$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_2)}{3!} f^{(3)}(c_x)$$

Differentiate the above, gives

$$f'(x) = P_2'(x) + D_x\left[\frac{(x-x_0)(x-x_1)(x-x_2)}{3!} f^{(3)}(c_x)\right]$$

$$= \frac{(2x-x_1-x_2)}{2h^2} f(x_0) - \frac{(2x-x_0-x_2)}{h^2} f(x_1) + \frac{(2x-x_0-x_1)}{2h^2} f(x_2)$$

$$+ \frac{(x-x_0)(x-x_1)+(x-x_2)(2x-x_0-x_1)}{6} f^{(3)}(c_x) + \frac{(x-x_0)(x-x_1)(x-x_2)}{3!} D_x\left[f^{(3)}(c_x)\right]$$

By taking $x = x_0$, $\quad f'(x_0) = \frac{-3}{2h} f(x_0) + \frac{2}{h} f(x_1) - \frac{1}{2h} f(x_2) + \frac{h^2}{3} f^{(3)}(c_x)$

By taking $x = x_2$, $\quad f'(x_1) = \frac{1}{2h} f(x_0) - \frac{2}{h} f(x_1) + \frac{3}{2h} f(x_2) + \frac{h^2}{3} f^{(3)}(c_x)$

By taking $x = x_1$, we have central difference formula.

Thus, we have

x	$f(x)$	$f'(x)$	Abs error	Error bound
$x_0 = 1.1$	9.025013	17.76971	0.280322	0.359033
$x_1 = 1.2$	11.02318	22.19364	0.147282	0.179517
$x_2 = 1.3$	13.46374	26.61757	0.309911	0.359033

21. Let $f(x) = x^4$

By trapezoidal rule, $\int_{0.5}^{1} x^4\,dx \approx \frac{0.5}{2}\left[f(0.5) + f(1)\right] = 0.265625.$

The error bound is $\left|\frac{h^3 f''(x)}{12}\right| = \left|\frac{0.5^3}{12} 12(1)^2\right| = 0.125.$

The exact solution is 0.071875, which is within the error bound.

By Simpson's rule $\int_{0.5}^{1} x^4\,dx \approx \frac{(0.25)}{3}\left[f(0.5) + 4f(0.75) + f(1)\right] = 0.1940104.$

The error bound is $\left|\frac{h^5 f^{iv}(x)}{90}\right| = \left|\frac{(0.25)^5}{90} 24\right| = 2.6042 \times 10^{-4}.$

The abs error is 2.604×10^{-4}, which is within the error bound.

22. Let $f(x) = x^2 e^{-x^2}$

(a) By composite trapezoidal rule

$$\int_0^2 x^2 e^{-x^2}\,dx \approx \frac{1}{2}\left[f(0) + 2f(1) + f(2)\right] = 0.404511.$$

By composite Simpson rule,

$$\int_1^2 x^2 e^{-x^2}\,dx \approx \frac{(0.5)}{3}\left[f(0) + 4f(0.5) + 2f(1) + 4f(1.5) + f(1)\right] = 0.422736.$$

(b) By composite trapezoidal rule,

$$\int_0^2 x^2 e^{-x^2}\, dx \approx \frac{0.25}{2}\left[f(0)+f(2)+2\left[f(0.25)+f(0.5)+f(0.75)+f(1)+f(1.25)+f(1.5)+f(1.75)\right]\right]$$

$$= 0.421582$$

By composite Simpson's rule,

$$\int_1^2 x^2 e^{-x^2}\, dx$$

$$\approx \frac{(0.25)}{3}\left[f(0)+4f(0.25)+2f(0.5)+4f(0.75)+2f(1)+4f(1.25)+2f(1.5)+4(1.75)+f(2)\right]$$

$$= 0.422716.$$

23. With n = 2 the composite trapezoidal rule gives

$$\int_0^1 f(x)\, dx \approx \frac{1}{4}\left[f(0)+2f(0.5)+f(1)\right] = 2 \qquad (1)$$

For n =4, the composite trapezoidal rule gives

$$\int_0^1 f(x)\, dx \approx \frac{1}{8}\left[f(0)+2f(0.25)+2f(0.5)+2f(0.75)+f(1)\right] = 1.75$$

Substituting (1) and $f(0.25) = f(0.75) = a$ gives

$$1.75 = \tfrac{1}{8}\left[8+4a\right]$$

Thus, a = 1.5.

24.

(a) Let $f(x) = e^x$

$$\int_{-1}^{1} e^x dx \approx f(-0.5773502692) + f(0.5773502692) = 2.342696.$$

(b) Let $f(x) = e^x$

$$\int_0^1 e^x dx = \frac{1}{2}\int_{-1}^{1} e^{\frac{1+t}{2}}\, dt \approx (0.555555556) f(-0.774596662) + 0.888888889 f(0)$$

$$+ (0.555555556) f(0.774596662) = 1.718281.$$

(c) Let $f(x) = (\cos x)^2$

$$\int_0^{\frac{p}{4}} (\cos x)^2\, dx = \frac{p}{8}\int_{-1}^{1}\left[\cos\frac{p}{8}(1+t)\right]^2 dt \approx (0.3478548451) f(-0.8611363116) + 0.6521451549 f(-0.3399810$$

$$+ (0.3478548451) f(0.8611363116) + 0.6521451549 f(0.3399810436)$$

$$= 1.718281.$$

25. Given $\int_0^1 \sqrt{x}\, dx = w_1 f(x_1)$ for $f(x)$ any linear polynomial.

Take $f(x) = 1$ $\quad \int_0^1 \sqrt{x}\, dx = w_1 \cdot 1$

Take $f(x) = x$, $\quad \int_0^1 \sqrt{x} \cdot x\, dx = \frac{2}{3} x_1$

Hence, $\quad \int_0^1 \sqrt{x}\; f(x)\, dx = \frac{2}{3} f\left(\frac{3}{5}\right)$

If $f(x) = e^{-x}$, then we have

$$\int_0^1 \sqrt{x}\; f(x)\, dx = \frac{2}{3} e^{-3/5} \approx 0.365874$$

26. Let $\int_0^1 f(x)\, dx = c_0 f(0) + c_1 f(x_1)$

Take $f(x) = 1$ $\quad \int_0^1 1\, dx = c_0 + c_1$

Take $f(x) = x$ $\quad \int_0^1 x\, dx = c_1 x_1$

Take $f(x) = x^2$ $\quad \int_0^1 x^2\, dx = c_1 x_1^2$

Solving the above, we have $c_0 = ¼$, $c_1 = ¾$, $x_1 = 2/3$.

27. Let $\int_{-1}^1 f(x)\, dx = af(-1) + bf(1) + cf'(-1) + df'(1)$

Take $f(x) = 1$. $\quad \int_{-1}^1 1\, dx = a + b = 2 \qquad (1)$

Take $f(x) = x$. $\quad \int_{-1}^1 x\, dx = -a + b + c + d = 0 \qquad (2)$

Take $f(x) = x^2$. $\quad \int_{-1}^1 x^2\, dx = a + b - 2c + 2d = \frac{2}{3} \qquad (3)$

Take $f(x) = x^3$. $\quad \int_{-1}^1 x^3\, dx = -a + b + 3c + 3d = 0 \qquad (4)$

Solving the above, we have $a = 1$, $b = 1$, $c = 1/3$, $d = -1/3$.

28. Let

$$f(x) = x^3 - 2x^2 - 5$$

The iterative formula is

$$x_{n+1} = x_n - \frac{(x_n^3 - 2x_n^2 - 5)}{3x_n^2 - 4x_n}$$

From the sketched graph, we choose x_0 = 2.5, then

n	x_n	$\lvert x_n - x_{n-1} \rvert$
0	2.5	
1	2.7142857	0.21429
2	2.6909515	0.02333342
3	2.6906475	0.000304
4	2.6906474	0.0000001 < 10^{-4}

Thus, $x \approx 2.6906474$.

29. In finding the root, we have

$$x = \sqrt[m]{a}$$
$$x^m = a$$

By solving $f(x) = x^m - a = 0$ would lead to the root as required.

The iterative formula is thus given by

$$x_{n+1} = x_n - \frac{\left(x_n^m - a\right)}{m x_n^{m-1}}$$
$$= \frac{1}{m}\left\{(m-1)x_n + \frac{a}{x_n^{m-1}}\right\}$$

To solve for $\sqrt{6}$, we have a = 6, m = 2. Taking the initial value 2.5, we have

n	x_n	$\lvert x_n - x_{n-1} \rvert$
0	2.5	
1	2.45	0.05
2	2.4494899	5.10×10^{-4}

Hence, $\sqrt{6} \approx 2.4494899$

30.

(a)

$$\left[\begin{array}{cc|c} 58.9 & 0.03 & 59.2 \\ -6.10 & 5.31 & 47.0 \end{array}\right]$$

$$m_{21} = -0.104, \left(r_2 - m_{21} r_1\right) \to r_2 \left[\begin{array}{cc|c} 58.9 & 0.03 & 59.2 \\ 0 & 5.31 & 53.2 \end{array}\right]$$

By back substitution, $x_1 = 1.00, x_2 = 10.0$.

(b)
(i)

$$\left[\begin{array}{ccc|c} 3.33 & 15900 & 10.3 & 7950 \\ 2.22 & 16.7 & 9.61 & 0.965 \\ -1.56 & 5.17 & -1.68 & 2.71 \end{array}\right]$$

$$\begin{array}{r} m_{21} = 0.666, (r_2 - m_{21} r_1) \to r_2 \\ m_{31} = -0.468, (r_3 - m_{31} r_1) \to r_3 \end{array} \left[\begin{array}{ccc|c} 3.33 & 15900 & 10.3 & 7950 \\ 0 & -10400 & 2.76 & -5280 \\ 0 & 7440 & 3.14 & 3720 \end{array}\right]$$

$$m_{32} = -0.715, (r_3 - m_{32} r_2) \to r_3 \left[\begin{array}{ccc|c} 3.33 & 15900 & 10.3 & 7950 \\ 0 & -10400 & 2.76 & -5280 \\ 0 & 0 & 5.11 & -50 \end{array}\right]$$

$$x_3 = -9.78, \quad x_2 = 0.504, \quad x_3 = 12.0$$

31.
(a)

$$\begin{aligned} x_1^{(k)} &= 0.9 + 0.1 x_2^{(k-1)} \\ x_2^{(k)} &= 0.7 + 0.1 x_1^{(k-1)} + 0.2 x_3^{(k-1)} \\ x_3^{(k)} &= 0.6 + 0.2 x_2^{(k-1)} \end{aligned}$$

k	0	1	2
$x_1^{(k)}$	0	0.9	0.97
$x_2^{(k)}$	0	0.7	0.91
$x_3^{(k)}$	0	0.6	0.74

(b)

$$\begin{aligned} x_1^{(k)} &= \tfrac{1}{3} + \tfrac{1}{3} x_2^{(k-1)} - \tfrac{1}{3} x_3^{(k-1)} \\ x_2^{(k)} &= \tfrac{-1}{2} x_1^{(k-1)} - \tfrac{1}{3} x_3^{(k-1)} \\ x_3^{(k)} &= \tfrac{4}{7} - \tfrac{3}{7} x_1^{(k-1)} - \tfrac{3}{7} x_2^{(k-1)} \end{aligned}$$

k	0	1	2
$x_1^{(k)}$	0	0.333 333	0.142 857
$x_2^{(k)}$	0	0	-0.357143
$x_3^{(k)}$	0	0.571 429	0.4285714

32.

(a)

$$x_1^{(k)} = 0.9 + 0.1x_2^{(k-1)}$$
$$x_2^{(k)} = 0.7 + 0.1x_1^{(k)} + 0.2x_3^{(k-1)}$$
$$x_3^{(k)} = 0.6 + 0.2x_2^{(k)}$$

k	0	1	2
$x_1^{(k)}$	0	0.9	0.979
$x_2^{(k)}$	0	0.79	0.9495
$x_3^{(k)}$	0	0.758	0.7899

(b)

$$x_1^{(k)} = \tfrac{1}{3} + \tfrac{1}{3}x_2^{(k-1)} - \tfrac{1}{3}x_3^{(k-1)}$$
$$x_2^{(k)} = \tfrac{-1}{2}x_1^{(k)} - \tfrac{1}{3}x_3^{(k-1)}$$
$$x_3^{(k)} = \tfrac{4}{7} - \tfrac{3}{7}x_1^{(k)} - \tfrac{3}{7}x_2^{(k)}$$

k	0	1	2
$x_1^{(k)}$	0	0.333 333	0.111 111
$x_2^{(k)}$	0	-0.166 667	-0.222 222
$x_3^{(k)}$	0	0.5	0.619 0477