MULTIMEDIA UNIVERSITY
Faculty of Engineering
PEM2056/PEM2062: Mathematics Techniques
Tutorial: Numerical Methods

1. Find the largest interval in which p^* must lie to approximate p with relative error at most
   (a) 10^{-3} for the value of p is 150 (b) 10^{-4} for the value of p is π.
   [(149.85, 150.15), (0.9999 π, 1.0001 π)]

2. Perform the following computation (i) exactly, (ii) using three-digit chopping arithmetic, and (iii) using three-digit rounding arithmetic. (iv) Compute the relative errors in parts (ii) and (iii).

   (a) $\left(\frac{1}{3}-\frac{3}{11}\right)+\frac{3}{20}$ (b) $\left(\frac{2}{3}+\frac{10}{11}\right)+\frac{3}{20}$
   [(a) 139/660, 0.211, 0.210, 2×10^{-3}, 3×10^{-3} (b) 1 479/660, 1.72, 1.73, 3×10^{-3}, 2×10^{-3}]

3. Suppose that $fl(y)$ is a k-digit rounding approximation to y. Show that

$$\left|\frac{y-fl(y)}{y}\right| \le 0.5\times10^{-k+1}$$

   [Hints: If $d_{k+1} < 5$, then $fl(y) = 0.d_1d_2\ldots d_k\times10^n$.
   If $d_{k+1} \ge 5$, then $fl(y) = 0.d_1d_2\ldots d_k\times10^n + 10^{n-k}$].

4. Consider a very limited binary normalized floating-point system in which there are four bits to store the positive numbers. Assume that the first digit of the mantissa is included.

   (a) What positive numbers can be represented if two bits are used for digits and two for exponents?
   (b) What positive numbers can be represented if three bits are used for digits and one for exponents?
   [(a) 1/2, ¾, 1, 1 ½, 2, 3, 4, 6, (b) ½, 5/8, ¾, 7/8, 1, 1 ¼, 1 ½, 1 ¾]

5. Define

$$f(x)=\frac{1-\cos(x)}{x^2}, \qquad g(x)=\frac{[\sin(x)/x]^2}{1+\cos x}, \qquad h(x)=2\left[\frac{\sin(\frac{x}{2})}{x}\right]^2.$$

   (a) Show that $f(x) = g(x) = h(x)$, if they exists for the x in the question.
   (b) Identify, for each of f, g and h, those ranges of x, if any, which lead to subtractive cancellation when they are evaluated as they stand. (1999)

6. In some situations, loss-of-significance errors can be avoided by rearranging the function begin evaluated. Do something similar for the following cases:

(a) $\sin(a + x) - \sin(a)$, x is near to 0
(b) $\log(x + 1) - \log(x)$, x large
(c) $\sqrt{9 + x^2} - 3$, if $|x|$ is small
(d) $\frac{1-\sqrt{1+x}}{\sqrt{x}}$, x is near to 0.

7. Suppose two points (x_0, y_0) and (x_1, y_1) are on a straight line with $y_1 \neq y_0$. Two formulas are available to find the x-intercept of the line:

$$x = \frac{x_0 y_1 - x_1 y_0}{y_1 - y_0}, \qquad \text{and} \qquad x = x_0 - \frac{(x_1 - x_0) y_0}{y_1 - y_0}$$

Use the data $(x_0, y_0) = (1.31, 3.24)$ and $(x_1, y_1) = (1.93, 4.76)$ and three-digit rounding arithmetic to compute the x-intercept both ways. Which method is better and why?

[(a) –0.00658, –0.0100 (b) second]

8. Construct the difference table

x	0	1	2	3	4	5	6	7
$y(x)$	0	0	0	1	1	0	0	0

Suppose that all $y(x)$ should be zero, but 1's are the errors. Observe the effect of adjacent "errors" of size 1.

$[\Delta^5 y : 5, 0, -5; \Delta^6 y : -5, -5]$

9. Compute the missing values in the following table.

$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$
.		
	.	
.		1
	.	
.		4
	5	
6		13
	.	
.		18
	.	
.		24
	.	
.		

[$f(x)$: 0, 0, 1, 6, 24, 60, 120]

10. Prove $\underset{h}{\Delta}(\log x_k) = \log\left(1 + \frac{h}{x_k}\right)$.

11. Use LIP of degrees 1, 2 and 3 to approximate $f(8.4)$ if $f(8.1) = 16.94410$, $f(8.3) = 17.56492$, $f(8.6) = 18.50515$, $f(8.7)=18.82091$.

[x=8.3, 8.6, $f(8.4)$=17.87833; x=8.3, 8.6, 8.7, $f(8.4)$=17.87716; x=8.1, 8.3, 8.6, 8.7,$f(8.4)$ = 17.87714]

If the data is generated using the function

$$f(x) = x \ln x.$$

Find a bound for the error and compare the bound to the actual error for the case of degree 2.

[1.367×10^{-5}, 1.452×10^{-5}]

12. Construct the interpolating polynomial for the unequally spaced points in the following table:

x	$f(x)$
0.0	–6.00000
0.1	–5.89483
0.3	–5.65014
0.6	–5.17788
1.0	–4.28172

If $f(1.1) = –3.99583$ is added to the above table, construct the new interpolating polynomial.

[$P_4(x) = -6 + 1.05170x + 0.57250x(x - 0.1) + 0.21500x(x-0.1)(x-0.3) + 0.063016x(x-0.1)(x-0.3)(x-0.6)$; $P_5(x) = P_4(x) + 0.014159x(x-0.1)(x-0.3)(x-0.6)(x-1)$.]

13. Complete the following divided-difference table. Hence, or otherwise, approximate $f(3.2)$.

x_k	$f[\,]$	$f[\,,\,]$	$f[\,,\,,\,]$	$f[\,,\,,\,,\,]$	$f[\,,\,,\,,\,,\,]$
1	0				
		341			
4	.		.		
		781		.	
3	.		.		.
		.		.	
–1	–2		50		
		11			
.	.				

[$f[\,,\,,\,]$: 38, 47; $f[\,,\,,\,,\,]$: 9; 336.3184]

14. Approximate $f(0.05)$ using the following data and the Newton forward-difference formula.:

x	0.0	0.2	0.4	0.6	0.8
$f(x)$	1.00000	1.22140	1.49182	1.82212	2.22554

Use the Newton backward-difference formula to approximate $f(0.65)$.
[1.05126, 1.91555]

15. Let $P_3(x)$ be the interpolating polynomial for the data (0,0), (0.5, y), (1,3) and (2,2). Find y if the coefficient of x^3 in $P_3(x)$ is 6. [4.25]

16. Find the degree of the interpolating polynomial that interpolates the following data.

x	-1	0	2	3	4	7
$f(x)$	6	5	21	38	61	166

If the above data is taken from a polynomial, what can you say about the degree of $f(x)$? Explain your answer. (2000)

17. Determine the error bound for linear polynomial interpolation in approximating $f(x) = \sin(x - 2)$ for an x in $\left[\frac{-\pi}{6}, \frac{\pi}{6}\right]$, Your answer should be in term of h, where $h = x_1 - x_0$.
[$h^2/8$]

18. Suppose you need to construct a table for the common logarithm function from x=1 to x=10 in such a way that linear interpolation is accurate to within 10^{-6}. Determine a suitable bound for the step size for this table. [4×10^{-3}]

19. By using forward or backward numerical differentiation, complete the following table:

x	$f(x)$	$f'(x)$
0.5	0.4794	
0.6	0.5646	
0.7	0.6442	

If the data were taken from $f(x) = \sin x$, compute the error bound and compare with the actual error.
[(0.852, 0.796, 0.796), error bound: 0.0282, 0.0322, 0.0322; or
(0.852, 0.852, 0.796), error bound: 0.0282, 0.0282, 0.0322]

20. By using the quadratic interpolating polynomial, determine the numerical differentiation formulas that can be used to approximate $f'(x)$ in the following table:

x	$f(x)$	$f'(x)$
1.1	9.025013	
1.2	11.02318	
1.3	13.46374	

Given that $f(x) = e^{2x}$, compute the actual error and compare with the error bound.
[(17.770, 22.194, 26.618); error bound: 0.359033, 0.179517, 0.359033]

21. Approximate

$$\int_{0.5}^{1} x^4 dx$$

by using the trapezoidal rule and Simpson's rule with one interval. Find a bound for the error for both method and compare to the actual error.
[0.265625, 0.071875, 0.125; 0.1940104, 2.604×10^{-4}, 2.6042×10^{-4}]

22. By using the composite trapezoidal rule and the composite Simpson's rule, approximate $\int_0^2 x^2 e^{-x^2}\, dx$ with

(a) 2 intervals (b) $h = 0.25$
[(0.404511, 0.422736), (0.421582, 0.422716)]

23. Suppose that $f(0.25) = f(0.75) = \alpha$. Find α if the Composite Trapezoidal rule with n=2 gives the value 2 for $\int_0^1 f(x)\, dx$ and with n=4 gives the value 1.75.
[1.5]

24. Approximate the following integrals using Gaussian quadrature:

(a) $\int_{-1}^{1} e^x\, dx$, with $n = 2$. (b) $\int_0^1 e^x\, dx$, with n=3
(c) $\int_0^{\pi/4} (\cos x)^2\, dx$, with n=4.
[2.342696, 1.718281, 0.642699]

25. Consider approximating integrals of the form

$$I(f)=\int_0^1 \sqrt{x}\, f(x)\,dx$$

in which $f(x)$ has several continuous derivatives on [0,1]. Find a formula

$$\int_0^1 \sqrt{x}\, f(x)\; dx \approx w_1 f(x_1)$$

which is exact if $f(x)$ is any linear polynomial.
Hence, find

$$\int_0^1 \sqrt{x}e^{-x}\; dx\ .$$

[2/3 f(3/5), 0.36587]

26. Find the constants c_0, c_1 and x_1 so that

$$\int_0^1 f(x)dx = c_0 f(0) + c_1 f(x_1)$$

has the highest possible degree of precision.

[1/4, 3/4, 2/3, 2]

27. Determine constants a, b, c and d that will produce a quadrature formula

$$\int_{-1}^1 f(x)dx = af(-1) + bf(1) + cf'(-1) + df'(1)$$

that has degree of precision 3. [1,1,1/3,-1/3]

28. Use Newtons's method to find solutions accurate to $|x_n - x_{n-1}| < 10^{-4}$ for $x^3 - 2x^2 = 5$, for x in the interval of [1,4].

[if p_0 = 2.5, p_4 = 2.69065]

29. Show that the iteration formula for Newton's Raphson method in solving the equation $f(x) = x^m - a = 0$ to determine the root $\sqrt[m]{a}$ is of the form

$$x_{n+1} = \frac{1}{m}\left\{(m-1)x_n + \frac{a}{x_n^{m-1}}\right\}.$$

Hence, use the iteration formula to solve for $\sqrt{6}$, to $|x_n - x_{n-1}| < 10^{-2}$.

[If x_0 = 2.5, x_2 = 2.4495]

30. Solve the following linear system.

(a) Perform your calculations in three-digit rounding arithmetic and by using Gauss elimination with partial pivoting.

$$\begin{aligned} 58.9x_1 + 0.03x_2 &= 59.2 \\ -6.10x_1 + 5.31x_2 &= 47.0 \end{aligned}$$

[10.0, 1.00]

(b) Perform your calculations in three-digit chopping arithmetic and by using Gauss elimination with
(i) partial pivoting,
(ii) scaled partial pivoting.

$$\begin{aligned} 3.3330x_1 + 15920x_2 + 10.333x_3 &= 7953, \\ 2.2220x_1 + 16.710x_2 + 9.6120x_3 &= 0.965, \\ -1.5611x_1 + 5.1792x_2 - 1.6855x_3 &= 2.714. \end{aligned}$$

[–9.78, 0.504, 12.0; –1.00, 0.500, 0.993]

31. Find the first two iterations of the Gauss-Jacobi method for the following systems, using $\mathbf{x}^{(o)} = \mathbf{0}$.

(a)

$$\begin{aligned} 10x_1 - x_2 &= 9, \\ -x_1 + 10x_2 - 2x_3 &= 7, \\ -2x_2 + 10x_3 &= 6. \end{aligned}$$

[$x_1^{(2)} = 0.97, x_2^{(2)} = 0.91, x_3^{(2)} = 0.74.$]

(b)

$$\begin{aligned} 3x_1 + 3x_2 + 7x_3 &= 4, \\ 3x_1 - x_2 + \mathrm{x}_3 &= 1, \\ 3x_1 + 6x_2 + 2x_3 &= 0. \end{aligned}$$

[$x_1^{(2)} = 0.1429, x_2^{(2)} = -0.3571, x_3^{(2)} = 0.4286.$]

32. Repeat exercise 34 with Gauss-Siedel method.

[$x_1^{(2)} = 0.979, x_2^{(2)} = 0.9495, x_3^{(2)} = 0.7899.$;

[(b) $x_1^{(2)} = 0.1111, x_2^{(2)} = -0.2222, x_3^{(2)} = 0.6190.$]