Solution to Assignment 2

Let

$$\begin{aligned} y_1 &= y \\ y_2 &= y' \\ y_3 &= y'' \end{aligned}$$

Then

$$\begin{aligned} y_1' &= y_2 \\ y_2' &= y_3 \\ y_3' &= 6y_3 - 11y_2 + 6y_1 \end{aligned}$$

In matrix form, we have

$$Y' = \begin{pmatrix} y_1' \\ y_2' \\ y_3' \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 6 & -11 & 6 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = AY .$$

The characteristic equation is

$$|A - lI| = 0$$

$$\begin{vmatrix} -l & 1 & 0 \\ 0 & -l & 1 \\ 6 & -11 & 6-l \end{vmatrix} = 0$$

or

$$l^3 - 6l^2 + 11l - 6 = 0$$

$$(l-1)(l-2)(l-3) = 0$$

$$\therefore \quad l = 1, 2, 3 .$$

When $l = 1$, $\underset{\sim}{e}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$; $l = 2$, $\underset{\sim}{e}_2 = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}$; $l = 3$, $\underset{\sim}{e}_3 = \begin{pmatrix} 1 \\ 3 \\ 9 \end{pmatrix}$. Therefore the nonsingular matrix P is $P = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{pmatrix}$ and $P^{-1}AP = D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$.

Let $Y = PU$, then $Y' = PU'$. Hence $Y' = AY \Rightarrow PU' = APU \Rightarrow U' = P^{-1}APU = DU$.

From $U' = \begin{pmatrix} u_1' \\ u_2' \\ u_3' \end{pmatrix} = DU = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$, $u_1' = u_1, u_2' = 2u_2, u_3' = 3u_3$.

Thus $u_1 = Ae^x$, $u_2 = Be^{2x}$, $u_3 = Ce^{3x}$.

From $Y = PU = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{pmatrix} \begin{pmatrix} Ae^x \\ Be^{2x} \\ Ce^{3x} \end{pmatrix}$, the solution to the 3rd order differential equation is

$$y = y_1 = Ae^x + Be^{2x} + Ce^{3x} .$$